Exam. Code : 103206
Subject Code : 1189

B.A./B.Sc. 6th Semester

MATHEMATICS

(Linear Algebra)

Paper—I

Time Allowed—2 Hours] [Maximum Marks—50

Note :— There are **EIGHT** questions of equal marks. Candidates are required to attempt any **FOUR** questions.

I. (a) Show that the set $G = \{i, -i, 1, -1\}$ is an abelian group with respect to multiplication.

(b) Show that the set of all polynomials with integral coefficients, is not a vector space over $\mathbb{R}$.

II. (a) Prove that the necessary and sufficient condition for a non-empty subset W of a vector space V(F) to be a subspace of V, is $\alpha x + \beta y \in W \ \forall \ \alpha, \beta \in F$ and $x, y \in W$.

(b) If V(F) be a vector space, prove that the set S of non-zero vectors $v_1, v_2, ..., v_n \in V$ is linearly dependent iff some element of S is a linear combination of the others.

III. (a) Prove that there exists a basis for each finite dimensional vector space.

(b) Find the co-ordinate vector of (1, 2, 1) relative to ordered basis B = {(2, 1, 0), (2, 1, 1), (2, 2, 1)}.

IV. (a) If U and W are two subspaces of a finite dimensional vector space V(F), prove that :

$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$

(b) Let W_1 and W_2 be two subspaces of $\mathbb{R}^4$;

$W_1 = \{(a, b, c, d) : b + c + d = 0\}$,

$W_2 = \{(a, b, c, d) : a + b = 0, c = 2d\}$

Find the basis and dimension of (i) W_1 (ii) W_2.

V. (a) Show that $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$T(x, y) = (x + 1, 2y, x + y)$

is not a linear transformation.

(b) Prove that every n-dimensional vector space over the field **F** is isomorphic to space $\mathbf{F}^n$.

VI. (a) If V(F) and W(F) are vector spaces over the same field F and $T : V \to W$ is a linear transformation. Prove that :

(i) Range of T is a subspace of W(F)

(ii) Null space of T is a subspace of V(F).

(b) For the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$ find basis and dimension of :

(i) Range space of T

(ii) Null space of T.

VII. (a) Let $B = \{v_1, v_2, ..., v_n\}$ be a basis for vector space V(F) and $T : V \to V$ be a linear transformation. Prove that for any

$v \in V$, $[T;B]\,[v;B] = [T(v);B]$

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$T(x, y, z) = (3x + z, -2x + y, -x + 2y + 4z)$

Find the matrix representation of T with respect to the basis B = {(1, 0, 1), (–1, 2, 1), (2, 1, 1)}.

VIII. (a) Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ determined by the matrix $M = \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 2 & 3 \end{bmatrix}$ with respect to ordered bases $B_1 = \{(1, 2), (0, 3)\}$ for $\mathbb{R}^2$ and $B_2 = \{(1, 1, 0), (0, 1, 1), (1, 1, 1)\}$ for $\mathbb{R}^3$.

(b) Let $B_1 = \{(1, 0), (0, 1)\}$ and $B_2 = \{(1, 3), (2, 5)\}$ be two ordered bases for $\mathbb{R}^2$. Find the transition matrix Q from B_2 to B_1 and the transition matrix P from B_1 to B_2. Also verify

$Q = P^{-1}$ and $P[v : B_2] = [v : B_1] \;\; \forall\, v \in \mathbb{R}^2$.